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Pseudo-Effect Algebras and Pseudo-Difference Posets

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In this paper, we introduce two different operations in pseudo-effect algebras and also introduce the pseudo-difference posets. We prove that the pseudo-effect algebras and the pseudo-difference posets are the same thing.

KEY WORDS: pseudo-effect algebras; difference operations; pseudo-difference posets.

1. EFFECT ALGEBRAS AND DIFFERENCE POSETS

Foulis and Bennet in 1994 introduced the following algebraic system $(E, \perp, \oplus, 0, 1)$ to model unsharp quantum logics, and $(E, \perp, \oplus, 0, 1)$ is said to be an *effect algebra* (Foulis and Bennet, 1994):

Let E be a set with two special elements 0, 1, $\perp$ be a subset of $E \times E$, if $(a, b) \in \perp$, denote $a \perp b$, and let $\oplus : \perp \to E$ be a binary operation, and the following axioms hold:

- (E1) (Commutative Law) If $a, b \in E$ and $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$.
- (E2) (Associative Law) If $a, b, c \in E, a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$, $a \perp (b \oplus c)$ and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
- (E3) (Orthocomplementation Law) For each $a \in E$ there exists a unique $b \in E$ such that $a \perp b$ and $a \oplus b = 1$.
- (E4) (Zero-Unit Law) If $a \in E$ and $1 \perp a$, then $a = 0$.

Let $(E, \perp, \oplus, 0, 1)$ be an effect algebra. If $a, b \in E$ and $a \perp b$ we say that a and b is *orthogonal*. If $a \oplus b = 1$ we say that b is the *orthocomplement* of a, and we write $b = a'$. Clearly $1' = 0$, $(a')' = a$, $a \perp 0$ and $a \oplus 0 = a$ for all $a \in E$.

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We say that $a \leq b$ if there exists $e \in E$ such that $a \perp e$ and $a \oplus e = b$. We may prove that $\leq$ is a partial ordering on E and satisfies that $0 \leq a \leq 1$, $a \leq b \Leftrightarrow b' \leq a'$ and $a \leq b' \Leftrightarrow a \perp b$ for $a, b \in E$.

If $a \leq b$ and $e \in E$ such that $a \perp e$, $a \oplus e = b$, we denote e as $b \ominus a$ and $\ominus$ is called the *difference operation*. Furthermore, it is easy to prove that if $a \leq b \leq c$, then the difference operation $\ominus$ satisfies the following properties (Foulis and Bennet, 1994):

(D1) $b \ominus a \leq b$.
(D2) $b \ominus (b \ominus a) = a$.
(D3) $(c \ominus b) \leq (c \ominus a)$.
(D4) $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

The properties D1–D4 are very important, in fact, Kopka and Chovanec in 1994 introduced the following quantum logic structure (Kopka and Chovanec, 1994):

A *difference poset* is a partially ordered set $(D, \leq, 0, 1)$ with a maximum element 1 and a minimum element 0, and with a partially defined binary operation $\ominus$ and $b \ominus a$ is defined iff $a \leq b$, and the operation $\ominus$ satisfies properties (D1)–(D4).

Thus, each effect algebra can induce a difference poset. Foulis and Bennet pointed out that the converse is also true (Foulis and Bennet, 1994). Thus, the effect algebras and the difference posets are the same thing.

2. PSEUDO-EFFECT ALGEBRAS AND DIFFERENCE OPERATIONS

By dropping the commutativity of effect algebras, Dvurecenskij and Vetterleinthat in 2001 introduced the new quantum logic structure and called it the *pseudo-effect algebra* (Dvurecenskij and Vetterleinthat, 2001):

Let PE be a set with two special elements 0, 1, $\perp$ be a subset of $PE \times PE$, if $(a, b) \in \perp$, denote $a \perp b$, and let $\oplus : \perp \to PE$ be a binary operation, and the following axioms hold:

(PE1) $a \oplus b$, $(a \oplus b) \oplus c$ exist iff $b \oplus c$, $a \oplus (b \oplus c)$ exist, and in this case, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(PE2) For each $a \in PE$, there is exactly one $d \in PE$, and exactly one $e \in PE$ such that $a \oplus d = e \oplus a = 1$.
(PE3) If $a \oplus b$ exists, there are elements $d, e \in PE$ such that $a \oplus b = d \oplus a = b \oplus e$.
(PE4) If $1 \oplus a$ or $a \oplus 1$ exist, then $a = 0$.

In view of (PE2), we may define the two unary operation $\sim$ and $-$ by requiring for any $a \in PE$,

$$a \oplus a^{\sim} = a^{-} \oplus a = 1.$$

Lemma 1. (Dvurecenskij and Vetterleinthat, 2001). *Let* $(PE, \oplus, \perp, 0, 1)$ *be a pseudo-effect algebra. For* $a, b, c \in PE$*, we have*

(i) $a \oplus 0 = 0 \oplus a = a$.
(ii) $a \oplus b = 0$ *implies that* $a = b = 0$.
(iii) $0^{\sim} = 0^{-} = 1, 1^{\sim} = 1^{-} = 0$.
(iv) $a^{\sim -} = a^{-\sim} = a$.
(v) $a \oplus b = a \oplus c$ *implies* $b = c$*, and* $b \oplus a = c \oplus a$ *implies* $b = c$ *(cancellation laws).*
(vi) $a \oplus c = b$ *iff* $a = (c \oplus b^{\sim})^{-}$ *iff* $c = (b^{-} \oplus a)^{\sim}$.
(vii) $a \oplus b$ *exists iff* $a \leq b^{-}$ *iff* $b \leq a^{\sim}$.

As the effect algebras manner, we can define a partial order for pseudo-effect algebras, that is, $a \leq b$ iff there exists some $c \in PE$, $a \perp c$ and $a \oplus c = b$.

It follows from (PE3) that $a \leq b$ iff there exists some $d \in PE$, $d \perp a$ and $d \oplus a = b$.

Now, we introduce two new operations $\ominus_l$ and $\ominus_r$ in pseudo-effect algebras as following:

Let $(PE, \oplus, \perp, 0, 1)$ be a pseudo-effect algebra, $a, b, c \in PE$. If $a \leq b$ and $c \oplus a = b$, we define c as the *left difference* of b and a, and denote $c = b \ominus_l a$. Dually, if $a \leq b$ and $a \oplus d = b$, we define d as the *right difference* of b and a, and denote $d = b \ominus_r a$.

It follows from Lemma 1 (v, vi) that the two operations $\ominus_l$ and $\ominus_r$ are well defined, and if $a \leq b$, then $(b \ominus_l a) = (a \oplus b^{\sim})^{-}$, $(b \ominus_r a) = (b^{-} \oplus a)^{\sim}$.

Theorem 1. *Let* $(PE, \oplus, \perp, 0, 1)$ *be a pseudo-effect algebra,* $a \leq b \leq c$*. Then we have*

(PD1) $b \ominus_l a \leq b, b \ominus_r a \leq b$.
(PD2) $b \ominus_l (b \ominus_r a) = a, b \ominus_r (b \ominus_l a) = a$.
(PD3) $(c \ominus_l b) \leq (c \ominus_l a), (c \ominus_r b) \leq (c \ominus_r a)$.
(PD4) $(c \ominus_l a) \ominus_r (c \ominus_l b) = b \ominus_l a, (c \ominus_r a) \ominus_l (c \ominus_r b) = b \ominus_r a$.
(PD5) *If* $1 \ominus_r (1 \ominus_l b \ominus_l a)$ *is defined, then there exist* $d, e \in PE$ *such that*

$$(1 \ominus_r (1 \ominus_l b \ominus_l a)) = (1 \ominus_r (1 \ominus_l a \ominus_l d)) = (1 \ominus_r (1 \ominus_l e \ominus_l b)).$$

If $1 \ominus_l (1 \ominus_r b \ominus_r a)$ *is defined, then there exists* $f, g \in PE$ *such that*

$$(1 \ominus_l (1 \ominus_r b \ominus_r a)) = (1 \ominus_l (1 \ominus_r a \ominus_r f)) = (1 \ominus_l (1 \ominus_r g \ominus_r b)).$$

Proof: (PD1) follows from the definitions of $\ominus_l$ and $\ominus_r$ immediately.

(PD2) It follows from the definitions of $\ominus_l$ and $\ominus_r$ that $(b \ominus_l a) \oplus a = b$, $a \oplus (b \ominus_r a) = b$, so $a = b \ominus_r (b \ominus_l a), a = b \ominus_l (b \ominus_r a)$.

(PD3) Let $c \ominus_l b = d$, $c \ominus_l a = e$, so $d \oplus b = c$, $e \oplus a = c$, and $d \oplus b = c = e \oplus a$. It follows from $a \leq b$ that there exists a $g \in PE$ such that $b = g \oplus a$, so $d \oplus (g \oplus a) = e \oplus a$. Note that $d \oplus (g \oplus a) = (d \oplus g) \oplus a$ and Lemma 1 (v) that we have $d \oplus g = e$, so $d \leq e$, that is, $(c \ominus_l b) \leq (c \ominus_l a)$. Dually, we may prove that $(c \ominus_r b) \leq (c \ominus_r a)$.

(PD4) Let $c \ominus_l b = d$, $c \ominus_l a = e$, so $d \oplus b = c$, $e \oplus a = c$. By $a \leq b$ that we can take a $g \in PE$ such that $b = g \oplus a$, so, $g = b \ominus_l a$. Since $d \oplus b = c = e \oplus a = d \oplus (g \oplus a) = (d \oplus g) \oplus a$, by Lemma 1 (v) again that $e = (d \oplus g)$, thus we have

$$(c \ominus_l a) \ominus_r (c \ominus_l b) = (e \ominus_r d) = g = (b \ominus_l a).$$

Dually, we get that

$$(c \ominus_r a) \ominus_l (c \ominus_r b) = b \ominus_r a.$$

(PD5) If $1 \ominus_r (1 \ominus_l b \ominus_l a)$ is defined, by Lemma 1 (vii) that $a \oplus b$ is also defined. It follows from the definition of pseudo-effect algebras that there exist $d, e \in PE$ such that $(a \oplus b) = (d \oplus a) = (b \oplus e)$, thus, we may prove that

$$(1 \ominus_r (1 \ominus_l b \ominus_l a)) = (1 \ominus_r (1 \ominus_l a \ominus_l d)) = (1 \ominus_r (1 \ominus_l e \ominus_l b)).$$

Dually, if $1 \ominus_l (1 \ominus_r b \ominus_r a)$ is defined, then there exists $f, g \in PE$ such that

$$(1 \ominus_l (1 \ominus_r b \ominus_r a)) = (1 \ominus_l (1 \ominus_r a \ominus_r f)) = (1 \ominus_l (1 \ominus_r g \ominus_r b)).$$

This theorem is proved. □

By using the two difference operations, we can easily show the relationship of effect algebras and pseudo-effect algebras, that is

Theorem 2. *Let $(PE, \oplus, \perp, 0, 1)$ be a pseudo-effect algebra. Then the following conditions are equivalent:*

(1) *PE is an effect algebra, that is, if $a \oplus b$ is defined, then $b \oplus a$ is also defined, and $(a \oplus b) = (b \oplus a)$.*
(2) *If $a \leq b$, then $(b \ominus_l a) = (b \ominus_r a)$.*
(3) *If $c \ominus_l a \ominus_l b$ is defined, then $c \ominus_l b \ominus_l a$ is also defined, and*

$$c \ominus_l a \ominus_l b = c \ominus_l b \ominus_l a.$$

(4) *If $c \ominus_r a \ominus_r b$ is defined, then $c \ominus_r b \ominus_r a$ is also defined, and*

$$c \ominus_r a \ominus_r b = c \ominus_l b \ominus_l a.$$

3. PSEUDO-DIFFERENCE POSETS

As the same as in effect algebras, the properties (PD1)–(PD5) are very important, in fact, we can introduce the following new quantum logic structure, we call it the *pseudo-difference poset*:

A pseudo-difference poset is a partially ordered set $(PD, \leq, 0, 1)$ with a maximum element 1 and a minimum element 0, two partial binary operations $\ominus_l$ and $\ominus_r$, and $b \ominus_l a$ is defined in PD iff $b \ominus_r a$ is defined in PD iff $a \leq b$ in PD, and the two operations $\ominus_l$ and $\ominus_r$ satisfy properties (PD1)–(PD5).

Theorem 1 showed that each pseudo-effect algebra can induce a pseudo-difference poset. Our Theorem 5 shows that the converse is also true. Thus, the pseudo-effect algebras and the pseudo-difference posets are the same thing.

At first, in a pseudo-difference poset $(PD, \leq, 0, 1)$, we can prove easily the following facts:

(1) $a \ominus_l 0 = a$, $a \ominus_r 0 = a$, for all $a \in PE$.
(2) $a \ominus_l a = 0$, $a \ominus_r a = 0$.
(3) If $a \leq b$, then $b \ominus_l a = 0$ iff $a = b$, $b \ominus_r a = 0$ iff $a = b$.
(4) If $a \leq b$, then $b \ominus_l a = b$ iff $a = 0$, $b \ominus_r a = b$ iff $a = 0$.
(5) If $c \ominus_l a = c \ominus_l b$, then $a = b$. If $c \ominus_r a = c \ominus_r b$, then $a = b$.
(6) If $a \ominus_l c = b \ominus_l c$, then $a = b$. If $a \ominus_r c = b \ominus_r c$, then $a = b$.

Furthermore, we prove the following interesting conclusion which shows that condition (PD4) can be substitute by other conditions, that is

Theorem 3. *Let $(PD, \leq, 0, 1)$ be a partially ordered set with a maximum element 1 and a minimum element 0, two partial binary operations $\ominus_l$ and $\ominus_r$, and $b \ominus_l a$ is defined in PD iff $b \ominus_r a$ is defined in PD iff $a \leq b$ in PD, and the two operations $\ominus_l$ and $\ominus_r$ satisfy condition (PD2). Then the condition (PD4) is equivalent to one of the following conditions:*

(PD6) $c \ominus_l a \ominus_r b = c \ominus_r b \ominus_l a$.
(PD7) $(c \ominus_l a) \ominus_l (b \ominus_l a) = (c \ominus_l b)$, $(c \ominus_r a) \ominus_r (b \ominus_r a) = (c \ominus_r b)$.

Proof: (PD4) $\Rightarrow$ (PD6). It follows from (PD2) that $a = (c \ominus_r (c \ominus_l a))$, so $(c \ominus_r b) \ominus_l a = (c \ominus_r b) \ominus_l (c \ominus_r (c \ominus_l a))$. Thus it follows from (PD4) that

$$(c \ominus_r b) \ominus_l a = (c \ominus_r b) \ominus_l (c \ominus_r (c \ominus_l a)) = (c \ominus_l a) \ominus_r b.$$

(PD6) $\Rightarrow$ (PD4). By (PD6) that $(c \ominus_l a) \ominus_r (c \ominus_l b) = (c \ominus_r (c \ominus_l b)) \ominus_l a$, so it follows from (PD2) that

$$(c \ominus_l a) \ominus_r (c \ominus_l b) = (c \ominus_r (c \ominus_l b)) \ominus_l a = b \ominus_l a.$$

Dually, we may prove that

$$(c \ominus_r a) \ominus_l (c \ominus_r b) = b \ominus_r a.$$

(PD4) ⇒ (PD7). It follows from (PD4) and (PD2) that

$$(c \ominus_l a) \ominus_l (b \ominus_l a) = (c \ominus_l a) \ominus_l ((c \ominus_l a) \ominus_r (c \ominus_l b)) = c \ominus_l b,$$

$$(c \ominus_r a) \ominus_r (b \ominus_r a) = (c \ominus_r b).$$

(PD7) ⇒ (PD4). It follows from (PD7) and (PD2) that

$$(c \ominus_l a) \ominus_r (c \ominus_l b) = (c \ominus_l a) \ominus_r ((c \ominus_l a) \ominus_l (b \ominus_l a)) = b \ominus_l a.$$

$$(c \ominus_r a) \ominus_l (c \ominus_r b) = (c \ominus_r a) \ominus_l ((c \ominus_r a) \ominus_r (b \ominus_r a)) = b \ominus_r a.$$

This completed the proof of Theorem 3. □

Theorem 4. *Let* $(PD, \leq, 0, 1)$ *be a pseudo-difference poset. Then* $(1 \ominus_l (1 \ominus_r a \ominus_r b))$ *exists iff* $(1 \ominus_r (1 \ominus_l b \ominus_l a))$ *exists and they are equal.*

Proof: We only need to prove they are equal. It follows from (PD6) and (PD2) that

$$(1 \ominus_l (1 \ominus_r a \ominus_r b)) \ominus_r a = (1 \ominus_r a \ominus_l (1 \ominus_r a \ominus_r b)) = b,$$

$$(1 \ominus_r (1 \ominus_l b \ominus_l a)) \ominus_l b = (1 \ominus_l b \ominus_r (1 \ominus_l b \ominus_l a)) = a.$$

So

$$(1 \ominus_l (1 \ominus_r a \ominus_r b)) \ominus_r a \ominus_l b = 0,$$

$$(1 \ominus_r (1 \ominus_l b \ominus_l a)) \ominus_l b \ominus_r a = 0.$$

By (PD6) again that

$$(1 \ominus_l (1 \ominus_r a \ominus_r b)) \ominus_r a \ominus_l b = 0,$$

$$(1 \ominus_r (1 \ominus_l b \ominus_l a)) \ominus_r a \ominus_l b = 0.$$

It follows from fact (5) and fact (6) that $(1 \ominus_l (1 \ominus_r a \ominus_r b)) = (1 \ominus_r (1 \ominus_l b \ominus_l a))$. The theorem is proved. □

Our mail result is:

Theorem 5. *Pseudo-effect algebras and pseudo-difference posets are the same thing.*

Proof: It follows from Theorem 1 that we only need to prove that each pseudo-difference poset may define a pseudo-effect algebra.

Let $(PD, \leq, 0, 1)$ be a pseudo-difference poset. Then we say that $b \oplus a$ is defined iff $a \leq 1 \ominus_r b$ and define the operation $\oplus$ as following:

$$(b \oplus a) := (1 \ominus_l (1 \ominus_r b \ominus_r a)).$$

Now, we show that $\oplus$ satisfies (PE1)–(PE4).

If $1 \oplus a$ is defined, then $(1 \oplus a) = (1 \ominus_l (1 \ominus_r 1 \ominus_r a))$, so $(1 \ominus_r 1 \ominus_r a)$ is also defined, that is, $0 \ominus_r a$ is defined. It follows from the definition of $\ominus_r$ that $a \leq 0$, so $a = 0$. Similar, if $a \oplus 1$ is defined, then $a = 0$. This showed that (PE4) holds.

Let $b \oplus a$ be defined, so $1 \ominus_l (1 \ominus_r b \ominus_r a)$ be also defined. It follows from (PD5) that there exist two elements $f, g \in PD$ such that

$$(1 \ominus_l (1 \ominus_r b \ominus_r a)) = (1 \ominus_l (1 \ominus_r a \ominus_r f)) = (1 \ominus_l (1 \ominus_r g \ominus_r b).$$

This showed that

$$(b \oplus a) = (a \oplus f) = (g \oplus b).$$

So (PE3) is proved.

For each $a \in PD$, $1 \ominus_r a$ exists and is unique, denote it as d. Note that $(1 \ominus_r a) \ominus_r d = 0$, $1 \ominus_l (1 \ominus_r a \ominus_r d) = 1$, it follows from the definition of $\oplus$ that $a \oplus d = 1$.

Dually, there exists a unique element e such that $e \oplus a = 1$. Thus, (PE2) is proved.

Let $y = (a \oplus b)$ and $z = (a \oplus b) \oplus c$ exist in PD. Then it follows from (PD4) and the definition of $\oplus$ that $(z \ominus_r a) \ominus_l (z \ominus_r y) = (y \ominus_r a)$, $y = (a \oplus b) = 1 \ominus_l (1 \ominus_r a \ominus_r b)$. Thus, it follows from the condition (PD6) of Theorem 3 and the condition (PD2) of pseudo-difference posets and Theorem 4 that

$$(y \ominus_r a) = (1 \ominus_l (1 \ominus_r a \ominus_r b) \ominus_r a) = ((1 \ominus_r a) \ominus_l (1 \ominus_r a \ominus_r b)) = b.$$

$$y = (a \oplus b) = (1 \ominus_r (1 \ominus_l b \ominus_l a)).$$

$$(y \ominus_l b) = (1 \ominus_r (1 \ominus_l b \ominus_l a) \ominus_l b) = (1 \ominus_l b \ominus_r (1 \ominus_l b \ominus_l a)) = a.$$

That is,

$$((a \oplus b) \ominus_l b) = a.$$

Now suppose $(w \ominus_l b) = a$, note that $((a \oplus b) \ominus_l b) = a = (w \ominus_l b)$, it follows from fact (6) that $w = (a \oplus b)$. Similar, if $h \ominus_r a = b$, then $h = (a \oplus b)$.

Furthermore, since $z = (y \oplus c)$, so use the same proof methods we may prove that $z \ominus_l c = y$. Thus, it follows from condition (PD6) that

$$((z \ominus_r a) \ominus_l c) = ((z \ominus_l c) \ominus_r a) = b.$$

On the other hand, it is easy to show that $c \leq 1 \ominus_r b$, so $b \oplus c$ exists. Note that $((z \ominus_r a) \ominus_l c) = b$ and $b \oplus c \ominus_l c = b$ we must have $z \ominus_r a = (b \oplus c) \in PD$.

Similar, we may prove that $a \oplus (b \oplus c)$ exists and $z = a \oplus (b \oplus c)$. Thus, (PE1) is also satisfied.

Finally, it is easy to show that the new left difference and right difference are induced by the pseudo-effect algebra which was defined as above and are just the same left difference and right difference of the pseudo-difference poset.

The theorem is proved. □

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REFERENCES

Dvurecenskij, A. and Vetterlein, T. (2001). Pseudo-effect algebras (I). *Basic properties*. *International Journal of Theoretical Physics* **40**, 685–701.

Foulis, D. J. and Bennett, M. K. (1994). Effect algebras and unsharp quantum logics. *Foundations of Physics* **24**, 1331–1352.

Kopka, F. and Chovanec, F. (1994). D posets. *Mathematics Slovaca* **44**, 21–34.